Gauge theories labelled by surgeries

Quantum Fields and Strings Group Journal Club and Seminar, BIMSA

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Outline

- DGG construction (hyperbolic)
- Surgeries and Kirby moves
- Matter circles
- Nahm sums
- Gauging and flips
- Fusion and descendent dualities
- Open problems

DGG construction

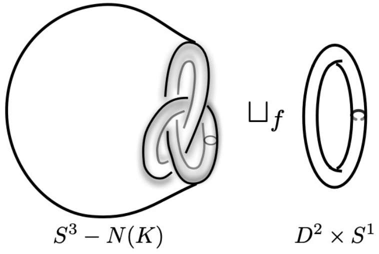
- Abelian 3d theories can be constructed by wrapping M5-branes on hyperbolic three-mfds.
- Hyperbolic mfds are knot complements with cusps which are singular in terms of metric, $S^3\backslash \mathrm{K}$, whose boundary is a torus T^2.
- Hyperbolic mfds can be interpreted as gluing tetrahedrons. Each chiral multiplets <-> tetrahedrons, superpotentials <-> gluing tetrahedrons. This correspondence is more like a conjecture from the perspective of 3d N=2 theories.
- The 3d theories realized by DGG is denoted by T[$S^3\backslash \mathrm{K}$], the 3-mfds are non-compact.

3d/3d and topological sector

- However, not all 3-mfds are hyperbolic. For instance, S^3 is one basic 3-mfds but not hyperbolic. So DGG construction should be a subset of the complect story.
- In recent years, Gukov and others consider the 3d-3d correspondence on compact mfds. The 3d Chern-Simons theroy on M_3 corresponds to a 3d N=2 theory $T[M_3]$. Both sides share the same 3-mfd.
- This means that we do not urgently need hyperbolic structures in many cases. We are free to consider topological structures and compact 3-mfds.

Surgeries and Kirby moves

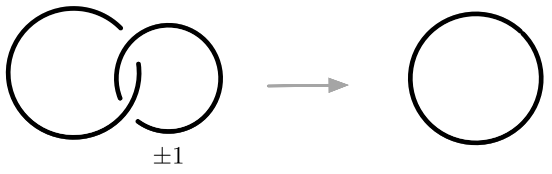
- To think about compact 3-mfds, we should rely on topological structrure, which is more basic and simple than hyperbolic. By topological structure, I means the one that is captured by surgeries and Kirby moves.
- Surgeries: we cut out the neighborhood of a link, and then glue back solid torus to each unknot component $S^3 \backslash L_1 \sqcup L_2 \sqcup L_3$.

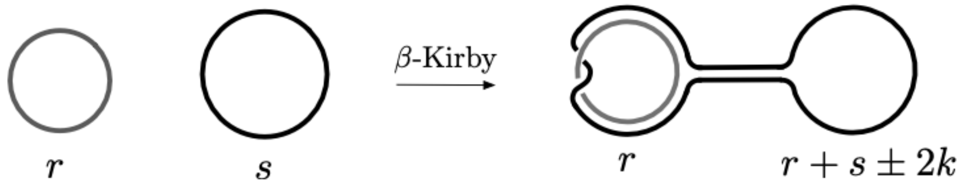


- The surgery along knots can be resolved into links.
- Surgery is an universal method to obtain all compact and oriented three-manifolds.

Kirby moves

- For a given three-manifold, the surgeries are not unique; there are many equivalent surgeries that are related through Kirby moves. Different surgeries are just different representations of the same 3-mfds.
- Kirby moves have two types: blow up/down, and handle slides.

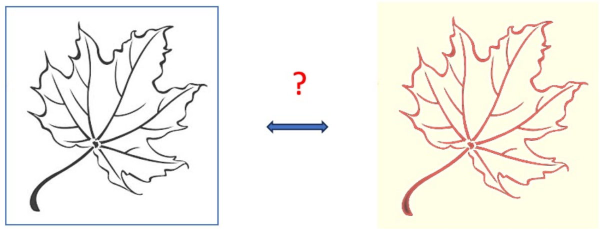




- In particular, there is one exceptional equivalent surgery called Rolfsen twist, which is an exception beyond Kirby moves, and particularly applies to identical surgery with infinite large framing number $f=\infty$. This twist could turn it into f =1.

Dualities

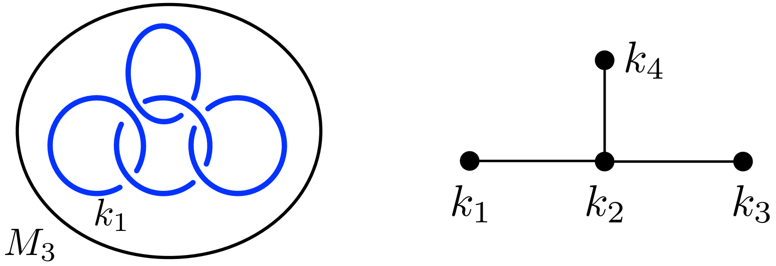
- Then, what do Kirby move mean in physics?
- If we believe the 3d theory $T[M_3]$ is determined by M_3 that plays the role of Seiberg-Witten curves, then the Kirby moves as the geometric structures of M_3 should be interpreted as dualities:



- The initial progress is in Gadde, Gukov, Putrov "Fivebranes and 4-mfds" [1306.4320]. 3d abelian theories are considered.
- In this paper the Kirby moves of first type are interpreted as integrating out/in gauge fields.

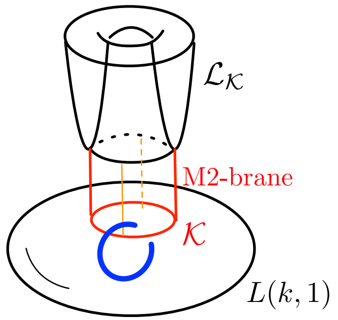
Dualities

- Moreover, linking numbers of surgery circles are interpreted as the mixed Chern-Simons levels:



- However, this story is not comple, as no coupled chiral multiplets is present in their theories. Then the question becomes how to produce matters through 3-mfds.
- Our claim is that the chiral multiplet can be realized by adding Ooguri-Vafa defects.

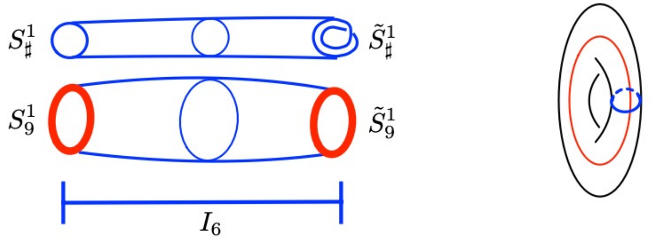
Ooguri-Vafa defects



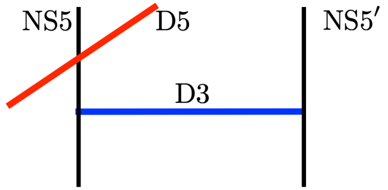
- Ooguri-Vafa defect is a Lagrangian submanifold in the cotangent bundle of the 3-mfds.
- The intersection between OV defects and 3-mfds are a circle. OV defects have the topology $R^2 \times S^1$. The intersection is $L \cap M_3 = S^1$.
- The OV defect is non-compact, and when we put a M5-brane on it, it gives a flavor symmetry $U(1)_F$.
- In order to know how this OV defect relates to matters, we need more detailed analysis on geometry.

Ooguri-Vafa defects

- As an insightful example, we think about the torus fiberation of Lens space L(k,1) , which is the gluing of two solid tori,



- The M/IIB duality maps it to NS5-D3-NS5 brane web in IIB, which realizes a pure 3d $U(1)_k$ theory. If we put OV defect along $S^1 \subset T^2$ which is longitude of the torus, then OV defect maps to a D5-brane which surely gives a hypermultiplet in 3d theroy.

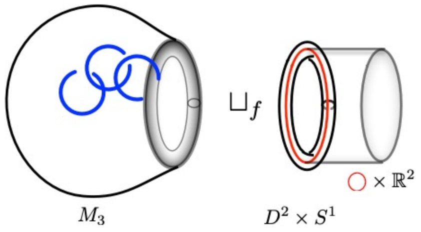


Matter circles and gauge circles

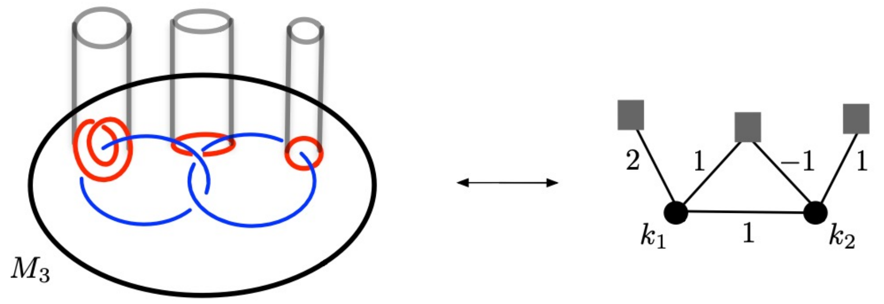
- We can propose a notation for two types of circles on a torus:

  Gauge circle (meridian) -> $U(1)_G$, matter circle (longitude) -> $U(1)_F$

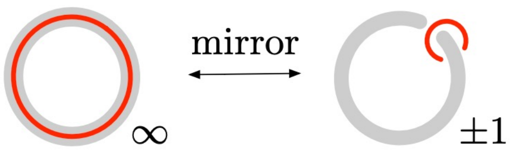
- The images of gauge circles are always surgery circles. The matter circles can be either on longitude of meridian, depending on framing numbers.
- What is interesting is the the linking number between gauge circle and matter circle is the charges of the matter.



- Example:



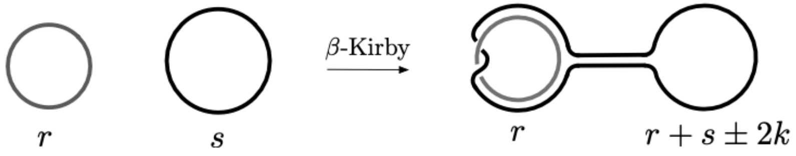
- We can use matter circles and gauge circles to represent a basic (ST-) duality $U(1)_{1/2} + 1\ \Phi$ <-> $1\ \Phi$, which is obtained by comparing Rolfsen twist with identical surgery.



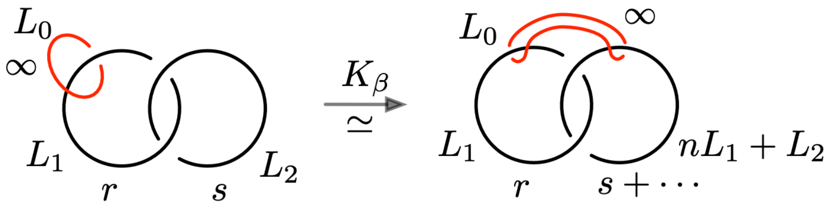
- ST-duality switches the locations of matter circles. ST-move is an extended Kirby move of blow up/down in the presence of OV defects.

Handle slides

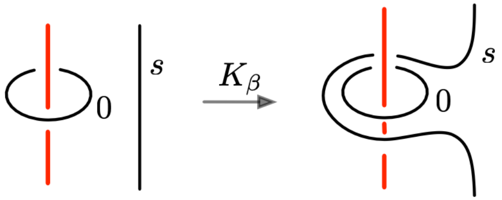
- We also have the handle slides:



- We find handle slides for guage circles are also preserved in the presence of OV defects, which basically change the locations of matter circles.



- One can even find an analogous figure below in 3-mfd textbook, which is caused by a handle slide:



Nahm sum

- Nahm sum is a nice form that the vortex partition functions (∼half-indices) could be written as:

$$P(C_{ij};x) = \sum_{n_1,n_2,\cdots,n_N=0}^{\infty} (-\sqrt{\mathfrak{q}})^{\sum_{i,j=1}^{N} C_{ij} n_i n_j} \prod_{i=1}^{N} \frac{x_i^{n_i}}{(\mathfrak{q},\mathfrak{q})_{n_i}} .$$

- We can slightly entend it by turning on mass parameters, generic charges, and adding one-loop contributions:

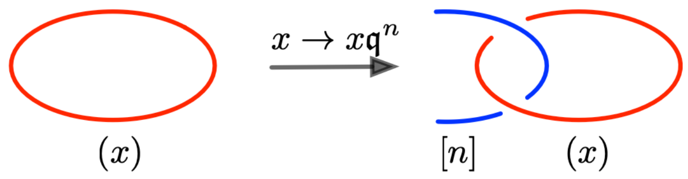
$$\begin{aligned} Z_{K_{ij}}(Q_\rho, x_i) &:= \prod_{\rho=1}^{N_f} (Q_\rho, \mathfrak{q})_\infty \cdot \sum_{n_1,\cdots,n_{N_c}=0}^{\infty} (-\sqrt{\mathfrak{q}})^{\sum_{i,j=1}^{N} K_{ij} n_i n_j} \frac{\prod_{i=1}^{N_c} x_i^{n_i}}{\prod_{\rho=1}^{N_f} (Q_\rho, \mathfrak{q})_{\sum_{i=1}^{N_c} q_i^\rho n_i^\rho}} \\ &= \sum_{n_1,\cdots,n_{N_c}=0}^{\infty} (-\sqrt{\mathfrak{q}})^{\sum_{i,j=1}^{N} K_{ij} n_i n_j} \prod_{i=1}^{N_c} x_i^{n_i} \prod_{\rho=1}^{N_f} \left(Q_\rho^{\sum_{i=1}^{N_c} q_i^\rho n_i^\rho}, \mathfrak{q} \right)_\infty , \end{aligned}$$

- Recently, I am aware that Nahm sum could describe the 3d N=4 rank 0 theories (ask Prof. Lee). Here, we will use it to read off geometries.

Gauge flavor symmetries

- The method of gauging can be read off from vortex partition functions:

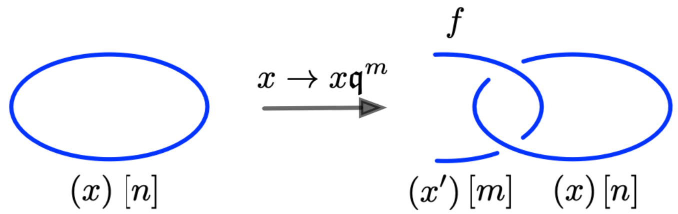
$$(x,\mathfrak{q})_\infty \xrightarrow{x\to x\mathfrak{q}^n} (x\mathfrak{q}^n,\mathfrak{q})_\infty = \frac{(x,\mathfrak{q})_\infty}{(x,\mathfrak{q})_n}$$



- Roughly, gauging a fundamental matter leads to a bi-fund. matter:

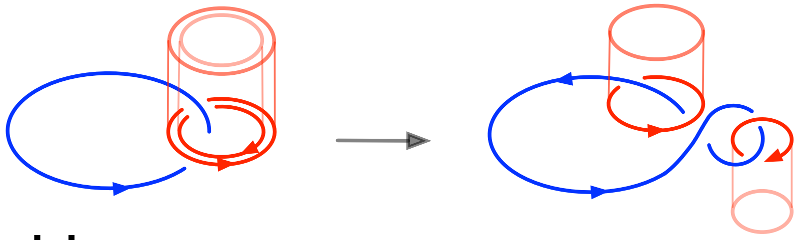
$$\bullet_{k_1} - \square \quad \longrightarrow \quad \bullet_{k_1} - \square - \bullet_{k_2}$$

- Gauging topological symmetry:



Flip mass parameters

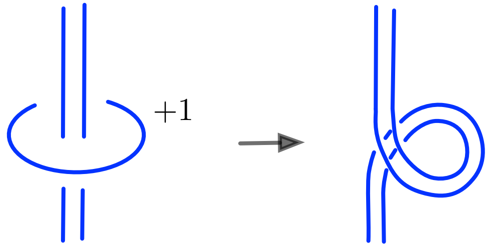
- Flipping the signs of mass parameters changes mixed Chern-Simons levels. Geometrically, it introduces a twist to gauge circles:



- The flips are described by

$$\frac{1}{(x,\mathfrak{q})_n(\mathfrak{q}x^{-1},\mathfrak{q})_{-n}} = \frac{1}{(-\sqrt{\mathfrak{q}})^{n^2}(x/\sqrt{\mathfrak{q}})^n}$$

- Generically, flipping a bi-fundamental matter is a Fenn-Rourke move:

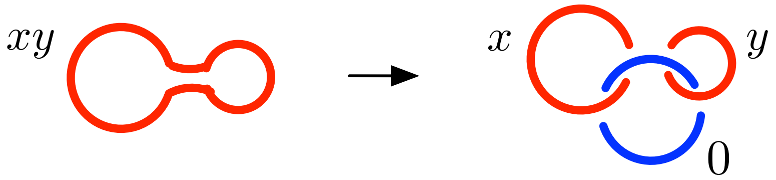


Fusion identity

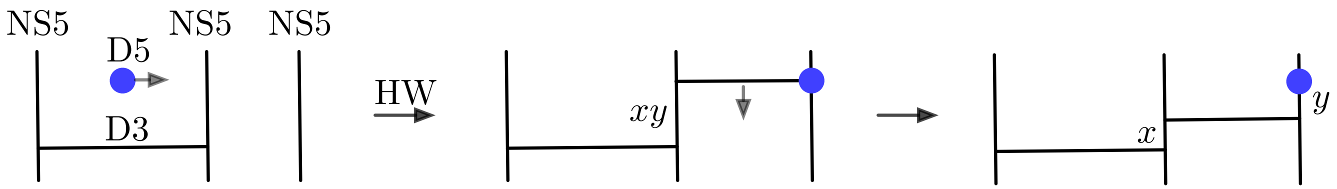
- We notice that there is a crucial identity that could derive all 3d abelian dualities that I know.

$$\frac{(xy,\mathfrak{q})_n}{(\mathfrak{q},\mathfrak{q})_n} = \sum_{k=0}^{n} \frac{(x,\mathfrak{q})_{n-k}}{(\mathfrak{q},\mathfrak{q})_{n-k}} \cdot \frac{(y,\mathfrak{q})_k}{(\mathfrak{q},\mathfrak{q})_k} \cdot x^k$$

- Fusion identity describes the connected sum of matter circles:

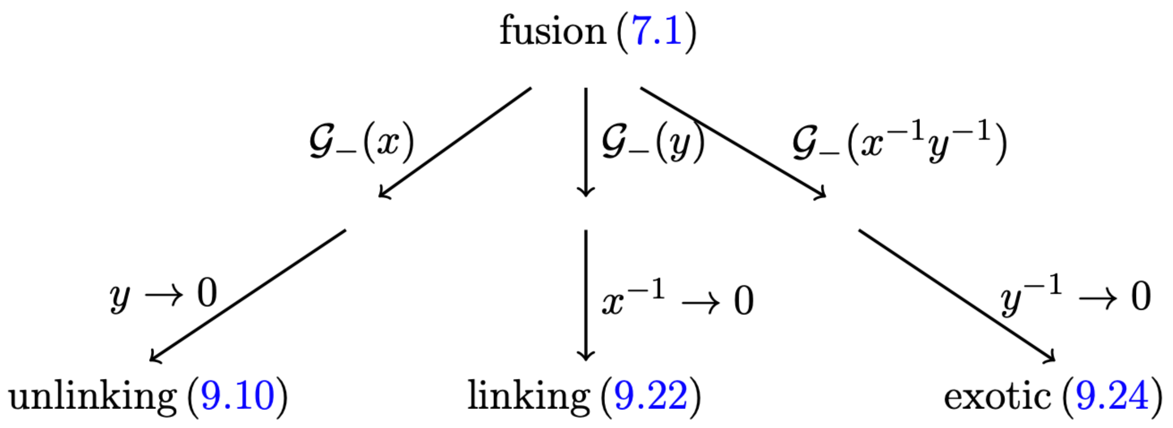


- Fusion identity is the operation of a gauging and a Hanany-Witten move:

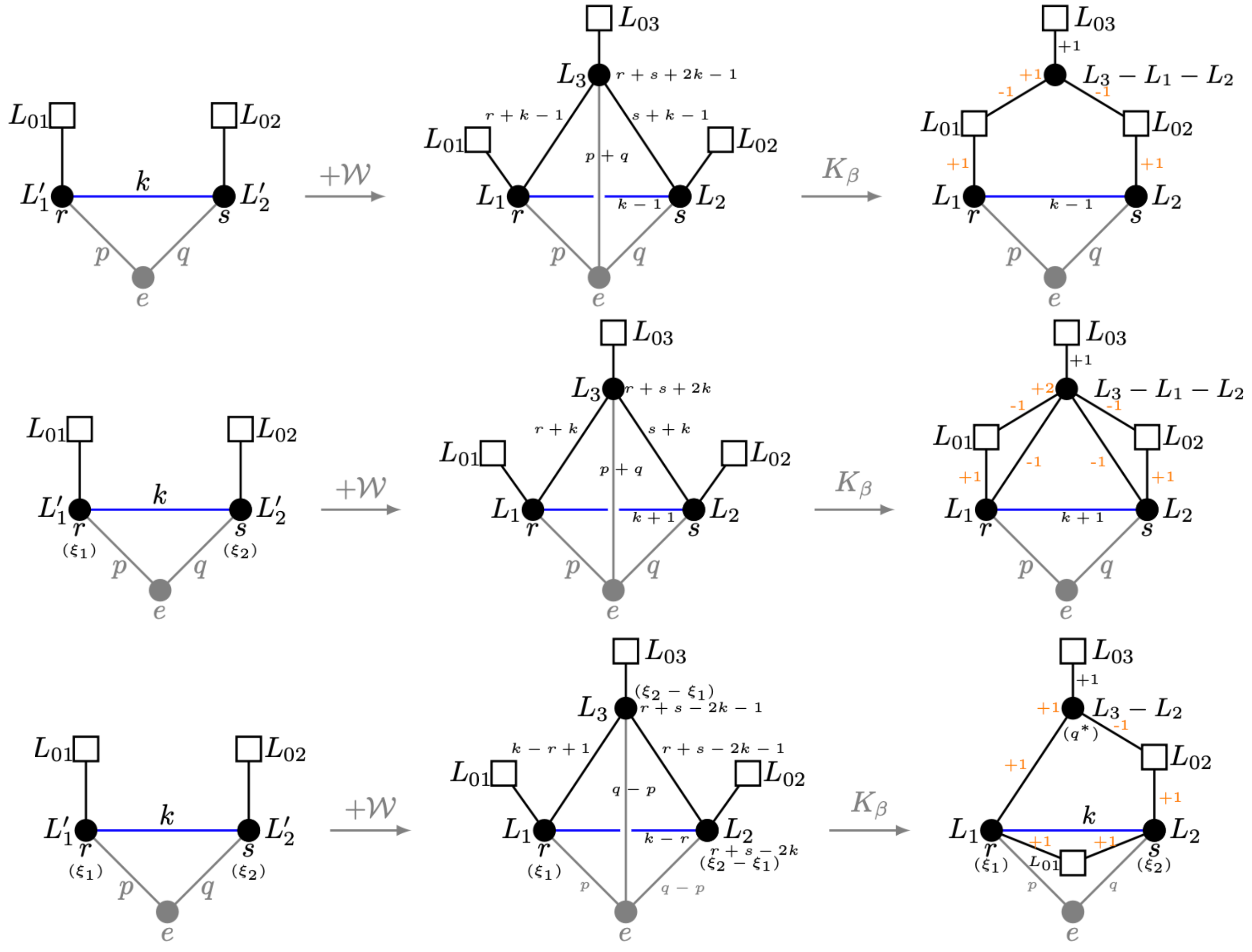


Fusion to superpotential

- Although fusion itself may not be an identity, but it derives the SQED-XYZ duality which contains a superpotential W=XYZ up to a gauging and a decoupling, so roughly fusion = superpotential.



Nahm sum for SQED-XYZ duality



$$\frac{(-\sqrt{\mathfrak{q}})^{2ab}x^{-b}}{(\mathfrak{q}x^{-1},\mathfrak{q})_a(\mathfrak{q},\mathfrak{q})_b} = \sum_{k=0}^{b} \frac{(-\sqrt{\mathfrak{q}})^{k^2}(1/\sqrt{\mathfrak{q}})^k}{(\mathfrak{q}x^{-1},\mathfrak{q})_{a-k}(\mathfrak{q},\mathfrak{q})_{b-k}(\mathfrak{q},\mathfrak{q})_k}$$

$$\frac{(-\sqrt{\mathfrak{q}})^{-2ab}y^{b}}{(\mathfrak{q}y^{-1},\mathfrak{q})_a(\mathfrak{q},\mathfrak{q})_b} = \sum_{k=0}^{b} \frac{(-\sqrt{\mathfrak{q}})^{2k^2-2ak-2bk}y^k}{(\mathfrak{q}y^{-1},\mathfrak{q})_{a-k}(\mathfrak{q},\mathfrak{q})_{b-k}(\mathfrak{q},\mathfrak{q})_k}$$

$$\frac{1}{(\mathfrak{q}\tilde{x},\mathfrak{q})_a(\mathfrak{q},\mathfrak{q})_b} = \sum_{k=0}^{b} \frac{(-\sqrt{\mathfrak{q}})^{k^2+2ak}(\sqrt{\mathfrak{q}}\tilde{x})^k}{(\mathfrak{q}\tilde{x},\mathfrak{q})_{a+b}(\mathfrak{q},\mathfrak{q})_{b-k}(\mathfrak{q},\mathfrak{q})_k}$$

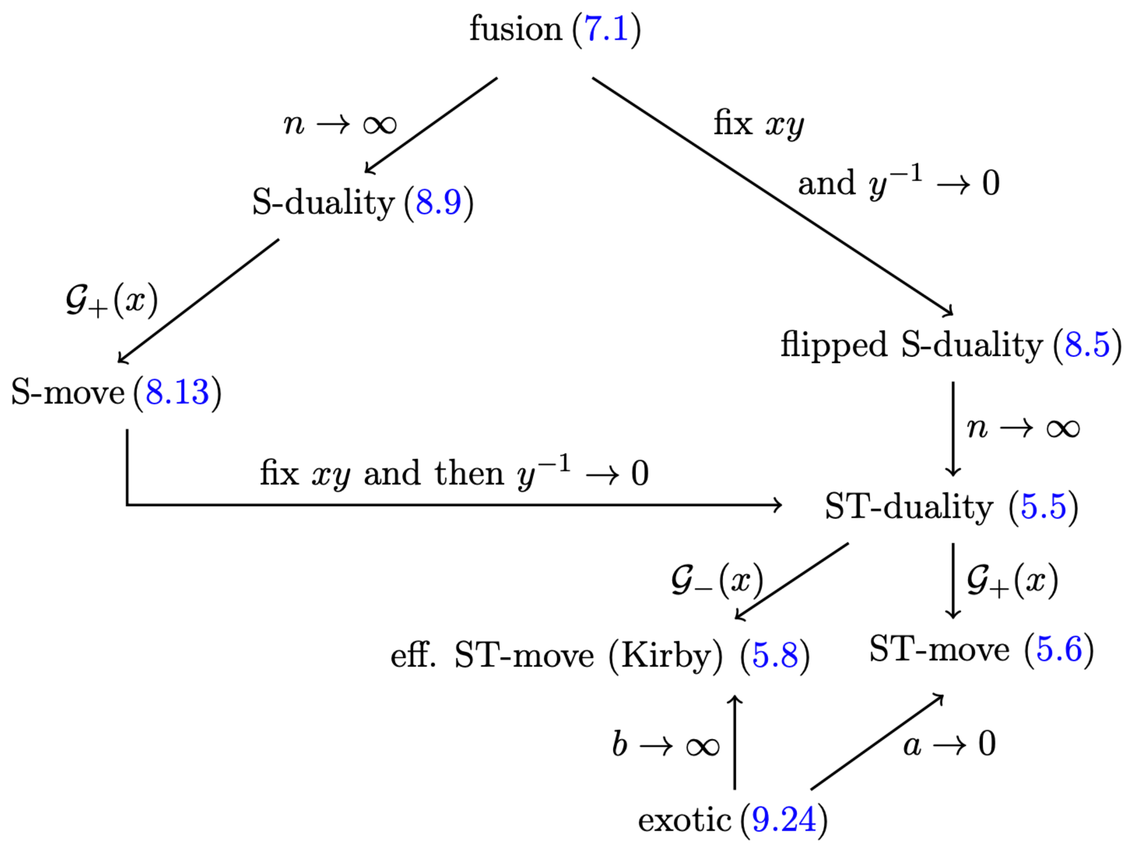
Descendent dualities

- Descendent dualities are described by non-trivial identities involving q-Pochhammer products, such as

$$(x,\mathfrak{q})_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (-\sqrt{\mathfrak{q}})^{j^2} (x/\sqrt{\mathfrak{q}})^j ,$$

$$\frac{(xy,\mathfrak{q})_\infty}{(x,\mathfrak{q})_\infty} = \sum_{k=0}^{\infty} \frac{(y,\mathfrak{q})_k}{(\mathfrak{q},\mathfrak{q})_k} \cdot x^k$$

$$\frac{1}{(x,\mathfrak{q})_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{(\mathfrak{q},\mathfrak{q})_n} \qquad (\tilde{x},\mathfrak{q})_\infty = \sum_{n=0}^{\infty} \frac{(\tilde{x}/\sqrt{\mathfrak{q}})^n}{(\mathfrak{q},\mathfrak{q})_n} \cdot (-\sqrt{\mathfrak{q}})^{n^2}$$



Hopf algebra (work in progress)

- The fusions and flips indicate a hiding algebraic structure, which may be a modified Hopf algebra of quantum groups.
- Hopf algebra is defied by properties:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$$

$$(m \circ (S \otimes \mathrm{id})) \circ \Delta = \epsilon = (m \circ (\mathrm{id} \otimes S) \circ \Delta)$$

- Roughly, antipode S =flip, and the third property is the fusion identity.
- This may imply the correspondence: 3d dualities <-> extended Kirby moves <-> modified Hopf algebra.

Open questions

- How to relate fusion to R-matrix
- 4d theory (2409.20393 “3D TFTs from 4d N = 2 BPS Particles”)
- Non-abelian theories.
- Many others

Thanks very much for your attention!